

UV caps and modulus stabilization for 6D gauged chiral supergravity

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ABSTRACT: We describe an explicit UV regularization of the brane singularities for all 4D flat configurations of 6D gauged chiral supergravity compactified on axially symmetric internal spaces (for which the general solutions are known). All such solutions have two or fewer co-dimension two singularities, which we resolve in terms of microscopic co-dimension one cylindrical 4-branes, whose interiors are capped using the most general possible 4D flat solution of the 6D field equations. By so doing we show that such a cap is always possible for any given bulk geometry, and obtain an explicit relationship between the properties of the capped 4-branes and the various parameters which describe the bulk solution. We show how these branes generically stabilize the size of the extra dimensions by breaking the scale invariance which relates classical solutions to 6D supergravity, and we compute the scalar potential for this modulus in the 4D effective theory. The lifting of this marginal direction provides a natural realization of the Goldberger-Wise stabilization mechanism in six dimensions.

KEYWORDS: Supergravity Models, Large Extra Dimensions, Flux compactifications.

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1. Introduction

Six-dimensional supergravity [1–4] has recently emerged as being a useful theoretical workshop within which to investigate phenomena which often generalize to systems having even more dimensions. Six dimensions are ideal for this purpose inasmuch as there are enough dimensions to permit the physics of most interest — such as chiral fermions [1], intricate Green-Schwarz anomaly cancellation [5] and flux-stabilized compactifications [1, 6]. Yet there are also few enough dimensions to allow the relevant field equations to be solved explicitly, allowing a detailed exploration of features which are more complicated to investigate in a 10- or 11-dimensional context.

Interest in six dimensions has been further sharpened by the recognition that it can provide significant insights into phenomenological problems in its own right. Prominent among these is the potential for having extra dimensions large enough to be relevant to precision measurements of gravity on micron length scales [7], and the potential of having the scale of gravity be as low as the weak scale [8]. Its supersymmetric version, with supersymmetry broken by branes, provides a realization of weak-scale supersymmetry breaking which does *not* predict the existence of superpartners for standard particles like the electron [9], and so whose implications for colliders differs considerably from standard

supersymmetric scenarios. It may yet provide an attractive approach to the cosmological constant problem [10]–[14], by building on the observation that higher-dimensional theories can break the link on which the cosmological constant problem rests: the link between the 4D vacuum energy density (which we believe to be large) and the curvature of 4D spacetime (which we observe to be small) [15] – [18].

The study of the physics of 6D supergravity was considerably advanced by the discovery of the most general class of compactifications to 4D flat space on an axially symmetric extra dimensional geometry [19, 11] which involve only the fields of the supergravity multiplet itself. Because these are the most general such solutions, they allow a more systematic study of the circumstances under which the observed, noncompact four dimensions are flat. In particular, these solutions are found to be singular at one or two locations within the extra dimensions [20], with the singularities being interpreted as representing the back-reaction of codimension-two 3-branes whose presence sources the fields described by the bulk fields under consideration. Of pressing interest is the identification of the kinds of brane properties which give rise to geometries with four flat observed dimensions.

Unfortunately, the characterization of the required brane properties is more complicated for codimension-two objects than it is for the more familiar codimension-one configurations familiar from Randall-Sundrum compactifications [21]. This is because the bulk fields sourced by higher codimension objects generically diverge at the positions of these objects. For this reason all detailed connections between bulk and brane properties have so far relied on the use of ‘thick’ branes — i.e. explicit models of the internal brane structure which allow the bulk-field singularities to be resolved, and smoothed out [22 – 25].

Our purpose in the present paper is to systematize this smoothing analysis to the general class of 4D flat solutions known for axially-symmetric internal geometries. We do so in order to provide a sufficiently general class of singularity resolutions to allow a meaningful mapping to be made between the properties of the resolved branes and those of the bulk geometries which they source. We resolve the bulk-field singularities at the source 3-branes by cutting off the bulk geometry with an explicit (but broad) class of cylindrical 4-branes which consistently couple to all of the relevant bulk fields. Their interiors are then capped off using the most general smooth, 4D-flat and cylindrically symmetry solutions to the same 6D supergravity equations as are solved by the bulk configurations.

Our main result is to provide explicit relations between the properties of the 4-branes (and their capped geometries) and those of the external bulk, a connection which pays at least two dividends.

- First, by sharpening the general relations between the brane and the bulk, our results provide the tools required to definitively explore the sensitivity of bulk properties to the UV structure on the source branes.
- Second, because the capped branes generically break the classical degeneracy between re-scaled bulk geometries, their presence lifts this degeneracy and so provides a stabilization mechanism which relates the size of the extra dimensions to properties of the source branes. This stabilization mechanism can be regarded as a particular form

of the general Goldberger-Wise mechanism [26] which arises particularly naturally within 6D supergravity.

Our presentation of these results proceeds as follows. Next, in section 2, we review the general 4D flat, cylindrically symmetric solutions of ref. [19], and use these to identify the form taken by the smooth geometries which cap the interiors of the cylindrical 4-branes. Section 3 then follows with a detailed discussion of the matching conditions which apply at the position of the 4-branes, and use these to identify the relationships which must exist between the parameters of the bulk solutions and those which govern the capped geometries and the intervening 4-branes. Section 4 then focusses on the implications of these relations for the parameters which govern the sizes of the bulk and capped geometries, and identify the choices which must be made on the branes in order to ensure a large hierarchy between the size of the bulk and the size of the ‘thick’ branes. Some conclusions are summarized in section 5.

2. Bulk solutions to 6D chiral supergravity

We next review the properties of the field equations of 6D gauged chiral supergravity [2–4], and present the most general solutions to these equations for which the induced geometry of the non-compact 4D directions is flat [19, 11, 20].

2.1 6D field equations

The action whose variation gives the field equations of interest is part of the Lagrangian density for 6D chiral gauged supergravity, and is given by¹

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2\kappa^2} g^{MN} \left[R_{MN} + \partial_M \phi \partial_N \phi \right] - \frac{1}{4} e^{-\phi} F_{MN} F^{MN} - \frac{2g^2}{\kappa^4} e^\phi, \quad (2.1)$$

where ϕ is the 6D scalar dilaton, and $F = dA$ is the field strength for the gauge potential, A_M , whose flux in the extra dimensions is what stabilizes the compactifications. The couplings g and κ have dimensions of inverse mass and inverse mass-squared, respectively. (We keep κ^2 explicit for ease of comparison with the various conventions which are used in the literature.)

These expressions set some of the bosonic fields of 6D supergravity to zero, as is consistent with the corresponding field equations (see however ref. [29] for solutions which do not make this assumption). The field equations for ϕ , A_M and g_{MN} are:

$$\begin{aligned} \square \phi + \frac{\kappa^2}{4} e^{-\phi} F_{MN} F^{MN} - \frac{2g^2}{\kappa^2} e^\phi &= 0 \\ D_M \left(e^{-\phi} F^{MN} \right) &= 0 \\ R_{MN} + \partial_M \phi \partial_N \phi + \kappa^2 e^{-\phi} F_{MP} F_N{}^P + \frac{1}{2} (\square \phi) g_{MN} &= 0. \end{aligned} \quad (2.2)$$

¹The curvature conventions used here are those of Weinberg’s book [27], and differ from those of MTW [28] only by an overall sign in the Riemann tensor.

The lagrangian density, eq. (2.1), has an important classical scaling property which plays a role in what follows: it re-scales as $\mathcal{L} \rightarrow e^{2\omega} \mathcal{L}$ when the fields undergo the constant re-scalings $g_{MN} \rightarrow e^\omega g_{MN}$, $e^\phi \rightarrow e^{\phi-\omega}$ and $A_M \rightarrow A_M$. Although it is not a symmetry of the action, it is a symmetry of the field equations and so its action always relates classical solutions to one another.

There is an ever-growing literature on the exact solutions to these equations, describing static compactifications of 6D down to 4D [2, 10, 11, 19, 20], as well as 4D de Sitter solutions [30], time-dependent solutions to the linearized equations [31, 32] and exact scaling solutions [33]. Our interest in what follows is in those which are cylindrically symmetric and asymptotically flat.

Boundary contributions. For later purposes we also record here the additional Gibbons-Hawking term [34] with which the above action must be supplemented when the field equations are investigated in the presence of boundaries. If the 6D spacetime region of interest, M , has a 5D boundary, $\Sigma = \partial M$, then the full action for the bulk fields is

$$S = \int_M d^6x \mathcal{L} - \int_\Sigma \sqrt{-\gamma} K, \tag{2.3}$$

where² γ_{mn} denotes the induced metric on Σ and $K = \gamma^{mn} K_{mn}$, is the trace of the extrinsic curvature tensor, K_{mn} , on Σ .

2.2 General bulk solutions

The most general axially-symmetric 4D-flat solutions to these bulk equations of motion are given by metrics of the form

$$ds^2 = e^{\omega-p} \mathcal{W}^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu + \mathcal{A}^2(\eta) \mathcal{W}^8(\eta) d\eta^2 + \mathcal{A}^2(\eta) d\psi^2, \tag{2.4}$$

where x^μ label the four noncompact dimensions, and $\{\eta, \psi\}$ are coordinates in the two extra dimensions, satisfying the periodicity condition $0 \leq \psi \leq 2\pi$. Solving the field equations, using for simplicity units for which $\kappa^2 = 1$, then gives³ the following formulae for the unknown functions $\mathcal{A}(\eta)$ and $\mathcal{W}(\eta)$ [19]

$$\begin{aligned} \mathcal{W}^4 &= \left| \frac{q\lambda_2}{2g\lambda_1} \right| \frac{\cosh[\lambda_1(\eta - \xi_1)]}{\cosh[\lambda_2(\eta - \xi_2)]} \\ \mathcal{A}^{-4} &= \left| \frac{2gq^3}{\lambda_1^3\lambda_2} \right| e^{-2(\lambda_3\eta+\omega)} \cosh^3[\lambda_1(\eta - \xi_1)] \cosh[\lambda_2(\eta - \xi_2)] \\ \text{while } e^{-\phi} &= \mathcal{W}^2 e^{\lambda_3\eta+\omega} \quad \text{and} \quad F_{\eta\psi} = \frac{q\mathcal{A}^2}{\mathcal{W}^2} e^{-\lambda_3\eta-\omega}. \end{aligned} \tag{2.5}$$

Here q , ω , λ_i ($i = 1, 2, 3$) and ξ_a ($a = 1, 2$) are arbitrary integration constants, subject only to the constraint $\lambda_2^2 = \lambda_1^2 + \lambda_3^2$. The role of the constant p is discussed further below.

²In the following we use capital latin letters for 6D indices (M, N) which run from $0 \dots 5$; lower-case latin letters for 5D indices (m, n) which run over the 4-brane directions, $0 \dots 4$; and greek letters (μ, ν) for 4D indices which run over the noncompact dimensions, $0 \dots 3$.

³Beware that ref. [19] instead uses $\kappa^2 = \frac{1}{2}$.

Notice that the signs of both λ_1 and λ_2 are irrelevant in these solutions, and so without loss of generality we take $\lambda_1 > 0$ and $\lambda_2 > 0$. Also, since in all subsequent equations it is only the magnitude of g which appears, we simplify notation by writing g instead of $|g|$.

For later convenience it is useful to display here the form of a gauge potential, A_M , whose differentiation gives the above field strength, $F_{\eta\psi}$:

$$A_\psi = \frac{\lambda_1}{q} \left(\tanh [\lambda_1 (\eta - \xi_1)] + \alpha \right), \tag{2.6}$$

where α is an arbitrary integration constant.

The parameters p , ω and ξ_1 . The parameters p and ω appearing in eq. (2.4), may appear unfamiliar to aficionados of ref. [19], since they are not seen in the solutions given there. They do not do so because each corresponds to a symmetry direction, and so for simplicity they are both removed in ref. [19]. We reinstate them here because we shall find that their removal is not similarly possible for the bulk *and* for the cap geometries which we consider shortly.

The symmetry corresponding to additive shifts of the variable ω is just the classical scale invariance of the field equations discussed above. The symmetry corresponding to p is similarly given by rigidly re-scaling the 4D metric, $g_{\mu\nu} \rightarrow e^{-p} g_{\mu\nu}$. This can be seen to be a symmetry of the field equations, eqs. (2.2), once these are restricted to the ansatz of eq. (2.4) together with $\phi = \phi(\eta)$ and $A_\psi = A_\psi(\eta)$. (Notice to this end that this ansatz implies in particular that the 4D part of the Ricci tensor, $R_{\mu\nu} = R^M{}_{\mu M \nu}$, scales in the same way as does the 4D metric, $g_{\mu\nu}$.)

There is a third parameter in eqs. (2.5), say ξ_1 , which could also have been eliminated in this way, since it can be removed by a suitable choice of the origin for the coordinate η . More formally, the field equations, eqs. (2.2), enjoy the symmetry $\eta \rightarrow \eta + \delta$, for constant δ , although the solutions, eqs. (2.5), do not. So applying such a shift to any given solution generates a one-parameter set of new solutions. In fact, inspection shows that the new solution obtained differs from the original one simply by making the changes

$$\xi_i \rightarrow \xi_i + \delta, \quad \omega \rightarrow \omega - \lambda_3 \delta \quad \text{and} \quad p \rightarrow p + \lambda_3 \delta. \tag{2.7}$$

This fact is important later since it tells us that one of the parameters which governs the bulk solutions can be arbitrarily removed by making an appropriate choice for the origin of coordinates for η . This means that one of these parameters, say ξ_1 , has no physical meaning and so one might wonder why we include it. The reason is that when branes are included, it is useful to use the η -shift symmetry to place them at convenient locations. Since we have then used up this symmetry, the parameter ξ_1 takes on a physical significance to do with the brane location.

Singularities. The bulk solutions of eqs. (2.5) are regular for all finite η , but generically are singular as $\eta \rightarrow \pm\infty$. The nature of these singularities is most easily seen by transforming to proper distance, $d\rho = \mathcal{A} \mathcal{W}^4 d\eta$. In this limit the extra-dimensional part of the metric becomes $d\rho^2 + C\rho^a d\psi^2$, which has a curvature singularity at $\rho \rightarrow 0$ provided $a \neq 2$.

If $a = 2$, the geometry has a conical singularity when $a = 2$ and $C \neq 1$. When $a = 2$ and $C = 1$ the solution is completely nonsingular at $\rho = 0$. (The only solution having no singularities at all is the Salam-Sezgin solution of ref. [1].)

Inspection of the asymptotic forms of eqs. (2.5) shows that both of the singularities (i.e. those at $\eta \rightarrow \pm\infty$) are conical if and only if $\lambda_1 = \lambda_2 \equiv \lambda$ (and so $\lambda_3 = 0$). For the 4D flat solutions considered here either both singularities are conical or neither of them are (see ref. [30] for non-flat solutions having only one conical singularity). When $\xi_1 \neq \xi_2$ the geometries with conical singularities are generically warped, giving the solutions of ref. [11]. However, if $\xi_1 = \xi_2$ the conical solutions degenerate into the unwarped ‘rugby ball’ solutions of ref. [10].

Physically, the singularities at $\eta \rightarrow \pm\infty$ indicate the presence of codimension-two source branes at these positions, with the singular behaviour arising because of the back-reaction of these branes onto the bulk fields. Furthermore, the precise kind of singularity is expected to be related to the properties of these source branes [35, 20, 33], with branes that source the dilaton field ϕ typically giving rise to a bulk scalar field configuration which diverges at the brane position, and so whose energy density can give rise to curvature singularities there.

Our goal in this section is to sharpen this connection, by relating more precisely the integration constants of the bulk solutions to the properties of the two source branes. We do so by explicitly resolving the singularities at $\eta \rightarrow \pm\infty$ in terms of a model of the microscopic structure of these two codimension-two branes.

2.3 Capped solutions

To this end we model each of the source branes as a cylindrical codimension-one 4-brane, situated at a fixed value of η , whose interior is filled in with one of the above bulk solutions that is nonsingular everywhere within the interior of the cylinder.

Consider then pasting together the following two metrics, along the 4+1 dimensional surface at $\eta = \eta_a$:

$$\begin{aligned} d\hat{s}^2 &= e^{\omega_a - p_a} \hat{\mathcal{W}}^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu + \hat{\mathcal{A}}^2(\eta) \hat{\mathcal{W}}^8(\eta) d\eta^2 + \hat{\mathcal{A}}^2(\eta) d\psi^2, & -\infty < \eta \leq \eta_a, \\ ds^2 &= e^\omega \mathcal{W}^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu + \mathcal{A}^2(\eta) \mathcal{W}^8(\eta) d\eta^2 + \mathcal{A}^2(\eta) d\psi^2, & \eta_a \leq \eta \leq \eta_b, \end{aligned}$$

with a similar splicing being performed at $\eta = \eta_b$ onto a nonsingular cap geometry which is defined for $\eta_b < \eta < \infty$. Codimension-one 4-branes will be located at the two boundaries $\eta = \eta_a$ and $\eta = \eta_b$, whose properties we determine below by using the appropriate jump conditions. Notice that we use the freedom to re-scale coordinates to set $p = 0$ in the bulk geometry (for $\eta_a < \eta < \eta_b$), but having done so we cannot also remove the dimensionless parameter p_a (or p_b) in the cap region.

For convenience we make here the choice that the coordinate location of the brane in the bulk coordinate system, η_a , is the same as its location in the cap coordinate system, $\hat{\eta}_a$. There is generically no reason for these two numbers to be the same, but as discussed earlier we may use the shift η -shift symmetry, eq. (2.7), to enforce $\eta_a = \hat{\eta}_a$. Having done this, we see that one of the previously unphysical parameters in the cap, say ξ_{1a} , takes on physical significance as it replaces $\hat{\eta}_a$.

For the cap solution which applies for $\eta < \eta_a$ we take one of the geometries of eqs. (2.5), subject to the condition that it be singularity free as $\eta \rightarrow -\infty$. This is only possible if it satisfies $\lambda_3 = 0$ — and so $\lambda_1 = \lambda_2 \equiv \lambda_a$ — leading to the form

$$\begin{aligned}
 e^{-\hat{\phi}} &= \hat{\mathcal{W}}^2 e^{\omega_a}, \\
 \hat{\mathcal{W}}^4 &= \left| \frac{q_a}{2g_a} \right| \frac{\cosh[\lambda_a(\eta - \xi_{1a})]}{\cosh[\lambda_a(\eta - \xi_{2a})]} \\
 \hat{\mathcal{A}}^{-4} &= \left| \frac{2g_a q_a^3}{\lambda_a^4} \right| e^{-2\omega_a} \cosh^3[\lambda_a(\eta - \xi_{1a})] \cosh[\lambda_a(\eta - \xi_{2a})] \\
 \hat{F}_{\eta\psi} &= \frac{q_a \hat{\mathcal{A}}^2}{\hat{\mathcal{W}}^2} e^{-\omega_a}.
 \end{aligned} \tag{2.8}$$

Similarly to the bulk case, we are free to take $\lambda_a > 0$. Also, as was done with g , for simplicity we write g_a in place of $|g_a|$. We are led in this way to the following 7 integration constants describing each capped geometry: $\lambda_a, p_a, q_a, \omega_a, \xi_{1a}, \xi_{2a}$ and η_a . By contrast, the constant g_a is *not* an integration constant, but is the $U_R(1)$ gauge coupling which appears in the bulk action whose equations of motion govern the solutions of interest. Although we keep g_a and g distinct in what follows, this is not crucial for our results, and one could instead choose to use the same action for the cap regions and the bulk between the two branes: $g_a = g$.

Requiring the cap geometry to be smooth for $\eta \rightarrow -\infty$ imposes the following relation amongst the integration constants:

$$|q_a| = 2\lambda_a g_a e^{\lambda_a(\xi_{2a} - \xi_{1a})}. \tag{2.9}$$

In what follows we regard this last equation as fixing the combination $\xi_{2a} - \xi_{1a}$. When the result satisfies $\xi_{1a} \neq \xi_{2a}$ the capped geometry is warped, and we refer to it as a ‘tear drop’. In the special case $\xi_{1a} = \xi_{2a} \equiv \xi_a$ — i.e. when $|q_a| = 2\lambda_a g_a$ — the cap geometry instead degenerates into a hemisphere.

Parameter counting. For future convenience it is useful at this point to count the number of integration constants associated with each of the solutions.

- *The Bulk:* Using the coordinate freedom to re-scale $g_{\mu\nu}$ and to shift η , we may set $p = 0$ and fix ξ_2 to a particular value. This leaves the general bulk solutions characterized by the 5 integration constants $\lambda_1, \lambda_2, \xi_2, q$ and ω .
- *The Caps:* The same coordinate freedom cannot again be used to similarly simplify the teardrop cap geometries for the regions $\eta < \eta_a$ and $\eta > \eta_b$. Once one parameter (e.g. ξ_{2a}) is used to ensure the cap geometry is everywhere smooth — c.f. eq. (2.9) — each cap is therefore described by 6 parameters. For the cap at $\eta < \eta_a$ these are $\lambda_a, \xi_{1a}, q_a, p_a$ and ω_a , together with the 4-brane location, η_a . For the cap at $\eta > \eta_b$ we instead have $\lambda_b, \xi_{1b}, q_b, p_b, \omega_b$ and η_b .

To these parameters we must also add those that characterize the 4-brane action, as is discussed in some detail in the next section.

We do not include the gauge potential integration constant, α , in the above counting because we handle its matching conditions separately in what follows. Besides α , the gauge potential also potentially hides other moduli describing how the background gauge field is embedded within the full gauge group. This can show up in the present analysis by making the gauge coupling constant, e , associated with the background gauge field potentially different from the coupling g which appears in the supergravity action, eq. (2.1), and so also in the solutions, eqs. (2.5) [10, 11].

3. Matching conditions

We next impose the matching conditions which apply across the 4-brane position, where the cap geometry meets that of the bulk. These come in two types: continuity of the fields g_{MN} , A_M and ϕ across $\eta = \eta_a$, and jump conditions which relate the discontinuity in the derivatives of these fields to properties of the 4-brane action.

3.1 Continuity conditions

Continuity of the bulk fields at each brane position provides 4 conditions among the parameters which define the caps. For instance, continuity across the 4-brane situated at η_a gives:

$$e^{\omega_a - p_a} \hat{\mathcal{W}}^2(\eta_a) = e^\omega \mathcal{W}^2(\eta_a), \quad \hat{\mathcal{A}}^2(\eta_a) = \mathcal{A}^2(\eta_a), \quad \hat{\phi}(\eta_a) = \phi(\eta_a) \quad (3.1)$$

and

$$\hat{A}_\psi(\eta_a) = A_\psi(\eta_a). \quad (3.2)$$

After some simplification, the three conditions of eqs. (3.1) reduce to the following relations amongst the parameters of the capped and bulk solutions

$$\frac{\cosh[\lambda_1(\eta_a - \xi_1)]}{\cosh[\lambda_a(\eta_a - \xi_{1a})]} = \left| \frac{\lambda_1 q_a}{\lambda_a q} \right| \quad (3.3)$$

$$\frac{\cosh[\lambda_2(\eta_a - \xi_2)]}{\cosh[\lambda_a(\eta_a - \xi_{2a})]} = \left| \frac{g_a \lambda_2}{g \lambda_a} \right| e^{2(\omega - \omega_a + \lambda_3 \eta_a)} \quad (3.4)$$

$$p_a = \lambda_3 \eta_a, \quad (3.5)$$

with a similar set of relations holding for brane b . As we see below in more detail in subsection (4.1), these equations can be regarded as fixing the three parameters p_a , ξ_{2a} and q_a , leaving λ_a , ω_a and η_a free.

Topological constraint. We treat the continuity condition for the gauge potential separately, because of a topological subtlety which arises in this case. Recall that the gauge potential for the bulk and capped regions can be written in the form

$$\begin{aligned} A_\psi &= \frac{\lambda_1}{q} \left(\tanh[\lambda_1(\eta - \xi_1)] + \alpha \right) & \eta_a < \eta < \eta_b \\ \hat{A}_\psi &= \frac{\lambda_a}{q_a} \left(\tanh[\lambda_a(\eta - \xi_{1a})] + 1 \right) & -\infty < \eta < \eta_a \end{aligned} \quad (3.6)$$

where the integration constant is chosen in the capped region to ensure that A_ψ vanishes as $\eta \rightarrow -\infty$, as is required for a nonsingular gauge potential. The same reasoning applied to the second capped region similarly gives

$$\begin{aligned} A_\psi &= \frac{\lambda_1}{q} \left(\tanh[\lambda_1(\eta - \xi_1)] + \alpha' \right) & \eta_a < \eta < \eta_b \\ \hat{A}_\psi &= \frac{\lambda_b}{q_b} \left(\tanh[\lambda_b(\eta - \xi_{1b})] - 1 \right) & \eta_b < \eta < \infty \end{aligned} \quad (3.7)$$

where the integration constant is in this case chosen in the capped region to ensure that A_ψ vanishes as $\eta \rightarrow +\infty$.

Naively we would determine α and α' by working within a gauge for which A_ψ is continuous for all η . However, the crucial point is that there is in general a topological obstruction to making such a choice for A_M everywhere. Instead we choose a gauge for which $A_\psi(\eta_a) = \hat{A}_\psi(\eta_a)$ and $A_\psi(\eta_b) = \hat{A}_\psi(\eta_b)$, and use these conditions to determine α and α' . But then α' and α cannot be taken to be equal on the region of overlap, $\eta_a < \eta < \eta_b$, but must differ instead by a gauge transformation. Following standard arguments, this leads to the quantization condition

$$\frac{\lambda_1}{q} (\alpha - \alpha') = \frac{N}{e} \quad (3.8)$$

where N is an integer, and e is the gauge coupling for the background gauge field (which need not equal g if the background flux is not the one gauging the specific $U_R(1)$ symmetry).

We find in this way that eq. (3.8) implies the following quantization condition on the various parameters:

$$\begin{aligned} \frac{N}{e} &= \frac{\lambda_1}{q} \left(\tanh[\lambda_1(\eta_b - \xi_1)] - \tanh[\lambda_1(\eta_a - \xi_1)] \right) \\ &\quad + \frac{\lambda_a}{q_a} \left(\tanh[\lambda_a(\eta_a - \xi_{1a})] + 1 \right) - \frac{\lambda_b}{q_b} \left(\tanh[\lambda_b(\eta_b - \xi_{1b})] - 1 \right). \end{aligned} \quad (3.9)$$

This generalizes to the case of thick branes the well-known Dirac quantization condition $N/e = 2\lambda_1/q$ [10, 36], which is retrieved from eq. (3.9) in the thin-brane limit obtained by taking $\eta_a \rightarrow -\infty$ and $\eta_b \rightarrow +\infty$.

Such arguments show that in general the continuity of the gauge potential across the two 4-branes, $\eta = \eta_a$ and $\eta = \eta_b$, determines the integration constants, α and α' which are specific to the gauge potentials. But the topological constraint then implies a single additional condition, eq. (3.9), which relates the bulk parameters, λ_1 , ξ_1 and q , to the undetermined brane quantities, η_a , ξ_{1a} , η_b , ξ_{1b} and the flux integer N .

3.2 Jump conditions

Having examined the continuity conditions, we next examine the relevant jump conditions which govern the discontinuity of derivatives of the bulk fields across the brane positions at $\eta = \eta_a$ and $\eta = \eta_b$. These junction conditions relate any such a discontinuity to the dependence of the intervening 4-brane action, \mathcal{S} , on these bulk fields, and may be derived

by integrating the equations of motion across a narrow interval around the 4-brane position: $\eta_a - \epsilon < \eta < \eta_a + \epsilon$, with ϵ taken negligibly small. Specialized to the metric these conditions are known as the Israel junction conditions [37].

One finds in this way

$$[\mathbb{K}_{mn}]_J = -T_{mn}, \quad [\sqrt{-g} e^{-\phi} F^{\eta m}]_J = -\frac{\delta \mathcal{S}}{\delta A_m} \quad \text{and} \quad [\sqrt{-g} \partial^\eta \phi]_J = -\frac{\delta \mathcal{S}}{\delta \phi}, \quad (3.10)$$

where we use the definition $[f(\eta)]_{\eta_a} \equiv f(\eta_a + \epsilon) - f(\eta_a - \epsilon)$. Here we define $K = \gamma^{mn} K_{mn}$ and $\mathbb{K}_{mn} = K_{mn} - \gamma_{mn} K$, where K_{mn} is the extrinsic curvature of the appropriate 4-brane surface.

4-Brane action. In order to proceed we require an ansatz for the 4-brane action. Consider therefore the following general choice

$$\mathcal{S} = - \int_{\Sigma} d^5 x \sqrt{-\gamma} \left[V(\phi) + \frac{1}{2} U(\phi) (D_m \sigma D^m \sigma) \right], \quad (3.11)$$

where γ_{mn} is the induced metric on the brane, and $V(\phi)$ and $U(\phi)$ are functions which determine the 4-brane couplings to the 6D dilaton.

Following ref. [22] we introduce a Stueckelberg field, σ , living on the brane, whose gauge covariant derivative is $D_m \sigma = \partial_m \sigma - e A_m$. We imagine this to be the low energy effective action obtained by integrating out the massive mode of some brane-localized Higgs field, $H = v e^{i\sigma}$, where v is an appropriate expectation value. Physically, this field describes supercurrents whose circulation can support changes in the background flux across the position of the 4-brane. (We return to the necessity for including such a field in subsequent sections.) The equation of motion for σ , together with the periodicity requirement $\psi \simeq \psi + 2\pi$, allows us to write the background configuration for σ as

$$\sigma = k \psi, \quad (3.12)$$

for some integer $k \in \mathbb{Z}$.

With these choices the jump conditions, eqs. (3.10), become

$$[\mathbb{K}_{\mu\nu}]_J = -T_{\mu\nu} \quad (3.13)$$

$$[\mathbb{K}_{\psi\psi}]_J = -T_{\psi\psi} \quad (3.14)$$

$$[\sqrt{-g} e^{-\phi} F^{\eta\psi}]_J = -e U \sqrt{-\gamma} D^\psi \sigma \quad (3.15)$$

$$[\sqrt{-g} \partial^\eta \phi]_J = \sqrt{-\gamma} \left[\frac{dV}{d\phi} + \frac{1}{2} (D_m \sigma D^m \sigma) \frac{dU}{d\phi} \right], \quad (3.16)$$

where the energy-momentum tensor derived from the above action is

$$\begin{aligned} T_{\mu\nu} &= -e^\omega \left(\frac{\mathcal{W}}{\mathcal{A}} \right)^2 \left[\mathcal{A}^2 V + \frac{1}{2} U (k - e A_\psi)^2 \right] \eta_{\mu\nu} \\ T_{\psi\psi} &= - \left[\mathcal{A}^2 V - \frac{1}{2} U (k - e A_\psi)^2 \right]. \end{aligned} \quad (3.17)$$

Here we see one reason for including the Stueckelberg field: without the function U the expressions for $T_{\mu\nu}$ and $T_{\psi\psi}$ are not independent since their ratio would be independent of parameters from the 4-brane action, leading to too restrictive a set of geometries which could be described in the bulk.

Evaluating the junction conditions. We next specialize the junction conditions to the explicit bulk fields discussed above. We first require the extrinsic curvature, K_{mn} , evaluated on both sides of the brane. In the bulk region, the unit normal to surfaces of constant η is

$$n_M = \mathcal{A}\mathcal{W}^4 \delta_M^\eta \tag{3.18}$$

and so the extrinsic curvature is given by $K_{mn} = \nabla_m n_n = -\mathcal{A}\mathcal{W}^4 \Gamma_{mn}^\eta$, where Γ_{mn}^η is the Christoffel symbol calculated from the full 6D metric. We find

$$\begin{aligned} \mathbb{K}_{\mu\nu} &= -\frac{e^\omega}{\mathcal{A}\mathcal{W}^2} \left[\frac{3\mathcal{W}'}{\mathcal{W}} + \frac{\mathcal{A}'}{\mathcal{A}} \right] \eta_{\mu\nu} \\ \mathbb{K}_{\psi\psi} &= -\frac{4\mathcal{A}\mathcal{W}'}{\mathcal{W}^5}, \end{aligned} \tag{3.19}$$

where primes denote differentiation with respect to η . Similarly, in the cap regions we have

$$\begin{aligned} \hat{\mathbb{K}}_{\mu\nu} &= -\frac{e^{\omega_a - p_a}}{\hat{\mathcal{A}}\hat{\mathcal{W}}^2} \left[\frac{3\hat{\mathcal{W}}'}{\hat{\mathcal{W}}} + \frac{\hat{\mathcal{A}}'}{\hat{\mathcal{A}}} \right] \eta_{\mu\nu} \\ \hat{\mathbb{K}}_{\psi\psi} &= -\frac{4\hat{\mathcal{A}}\hat{\mathcal{W}}'}{\hat{\mathcal{W}}^5}. \end{aligned} \tag{3.20}$$

- Evaluating the $(\mu\nu)$ Israel junction condition at $\eta = \eta_a$ then gives⁴

$$\begin{aligned} &\left(\frac{\lambda_3}{2} + e^{-2(\omega - \omega_a + \lambda_3 \eta_a)} \lambda_a \tanh[\lambda_a(\eta_a - \xi_{2a})] - \lambda_2 \tanh[\lambda_2(\eta_a - \xi_2)] \right) \\ &= -\mathcal{W}^4 \left[(\mathcal{A}V_a) + \frac{1}{2} \left(\frac{U_a}{\mathcal{A}} \right) (k_a - eA_\psi)^2 \right] \end{aligned} \tag{3.21}$$

where the subscript ‘a’ on V , U and k denotes the corresponding 4-brane property specialized to the brane at $\eta = \eta_a$.

- The $(\psi\psi)$ Israel junction condition similarly gives

$$\begin{aligned} &\left[\lambda_1 \tanh[\lambda_1(\eta_a - \xi_1)] - \lambda_2 \tanh[\lambda_2(\eta_a - \xi_2)] - e^{-2(\omega - \omega_a + \lambda_3 \eta_a)} \left(\lambda_a \tanh[\lambda_a(\eta_a - \xi_{1a})] \right. \right. \\ &\quad \left. \left. - \lambda_a \tanh[\lambda_a(\eta_a - \xi_{2a})] \right) \right] = -\mathcal{W}^4 \left[(\mathcal{A}V_a) - \frac{1}{2} \left(\frac{U_a}{\mathcal{A}} \right) (k_a - eA_\psi)^2 \right]. \end{aligned} \tag{3.22}$$

Taking the sum and the difference of these last two conditions allows the isolation of conditions for V_a and U_a separately. It is also easy to see that the resulting equations always admit real solutions for any value of the bulk parameters and the brane position.

- The junction condition for the gauge field similarly evaluates to

$$q - q_a e^{-2(\omega - \omega_a + \lambda_3 \eta_a)} = -e \mathcal{W}^4 \left(\frac{U_a}{\mathcal{A}} \right) (k_a - eA_\psi). \tag{3.23}$$

Notice that we can eliminate the two brane quantities, U_a and V_a , from the previous three jump conditions to obtain a constraint that does not depend on 4-brane parameters.

⁴It is understood in what follows that all functions depending on η are evaluated at $\eta = \eta_a$.

Indeed, by subtracting eq. (3.21) from eq. (3.22), and then dividing the result by eq. (3.23), we obtain the expression

$$\frac{\frac{1}{2}\lambda_3 + \lambda_a \tanh[\lambda_a(\eta_a - \xi_{1a})] - \lambda_1 \tanh[\lambda_1(\eta_a - \xi_1)]}{e^{-2(\omega - \omega_a + \lambda_3 \eta_a)} q_a - q} = -\frac{k_a}{e} + \frac{\lambda_a}{q_a} \left(\tanh[\lambda_a(\eta_a - \xi_{1a})] + 1 \right). \quad (3.24)$$

An identical argument for brane b similarly gives:

$$\frac{\frac{1}{2}\lambda_3 + \lambda_b \tanh[\lambda_b(\eta_b - \xi_{1b})] - \lambda_1 \tanh[\lambda_1(\eta_b - \xi_1)]}{e^{-2(\omega - \omega_b + \lambda_3 \eta_b)} q_b - q} = -\frac{k_b}{e} + \frac{\lambda_b}{q_b} \left(\tanh[\lambda_b(\eta_b - \xi_{1b})] - 1 \right). \quad (3.25)$$

• By contrast, the dilaton junction condition gives a condition on the ϕ -derivatives of U_a and V_a :

$$2\lambda_3 + \lambda_1 \tanh[\lambda_1(\eta_a - \xi_1)] - \lambda_2 \tanh[\lambda_2(\eta_a - \xi_2)] - e^{-2(\omega - \omega_a + \lambda_3 \eta_a)} \left(\lambda_a \tanh[\lambda_a(\eta_a - \xi_{1a})] - \lambda_a \tanh[\lambda_a(\eta_a - \xi_{2a})] \right) = -2\mathcal{W}^4 \left[\mathcal{A} \frac{dV_a}{d\phi} + \frac{1}{2\mathcal{A}} \frac{dU_a}{d\phi} (k_a - eA_\psi)^2 \right], \quad (3.26)$$

which, using the $(\psi\psi)$ Israel jump condition, simplifies to

$$2\lambda_3 = \mathcal{W}^4 \left[\mathcal{A} \left(V_a - 2 \frac{dV_a}{d\phi} \right) - \frac{1}{2\mathcal{A}} \left(U_a + 2 \frac{dU_a}{d\phi} \right) (k_a - eA_\psi)^2 \right]. \quad (3.27)$$

Conditions for scale invariance. Before proceeding it is useful to pause at this point to record the unique choice for the functions V_a and U_a which preserves the classical scaling symmetry of the bulk equations of motion, corresponding to the transformation $\omega \rightarrow \omega + \Delta$ and $\omega_a \rightarrow \omega_a + \Delta$.

Inspection shows that the continuity equations remain unchanged by this transformation because ω and ω_a only appear there in the combination $\omega - \omega_a$. The left-hand-sides of the various jump conditions remain similarly unchanged. On the right-hand-side, however, we see that \mathcal{A} transforms, and so invariance requires $V_a(\phi)$ and $U_a(\phi)$ to transform in a way which cancels the transformation of \mathcal{A} . Such an invariant choice for U_a and V_a is possible for the Israel and Maxwell jump conditions, eqs. (3.21), (3.22), and (3.23), because within these U_a and V_a only appear with \mathcal{A} in the combinations $\mathcal{A}V_a$ and U_a/\mathcal{A} . It follows that preservation of the classical scaling symmetry requires

$$V_a = v_a e^{\phi/2} \quad \text{and} \quad U_a = u_a e^{-\phi/2}, \quad (3.28)$$

in agreement with the analysis of ref. [11]. Any other choices for these functions necessarily breaks the classical scale invariance of the problem.

It then remains to determine what invariance requires for the dilaton jump condition, eq. (3.27). When this is specialized to the scale invariant case, eqs. (3.28), the right-hand side degenerates to zero, giving the simple condition $\lambda_3 = 0$. Besides imposing no new conditions on U_a and V_a , this tells us that scale-invariant brane configurations can only source bulk geometries satisfying $\lambda_3 = 0$, and hence only having conical singularities. Since all of the geometries having two conical singularities are 4D flat [30], we see in detail how the jump conditions enforce the connection between scale invariance and 4D flatness.

4. Applications

Given the general bulk and cap solutions, and a complete set of matching conditions, we may now see what the solutions to these conditions tell us about bulk-brane dynamics in six dimensions. In this section we use the above formalism to address two questions. First: given a bulk geometry what kinds of caps are possible? Second: given specific brane properties, what kinds of bulk are generated? In particular, in this second case we ask how the breaking of scale invariance by the branes can lead to the stabilization of the extra-dimensional size.

Answering this last question allows us also to address an issue of potential importance for phenomenology: what conditions must the cap and bulk parameters satisfy in order to have a large hierarchy between the volumes of the caps and the volume of the bulk? This point is important when the regularizing 4-branes and caps are regarded as specifying the microscopic structure of 3-branes that sit at the singular points of the geometry.

4.1 Capping a given bulk

We begin by studying what kinds of caps can be used to smooth a generic bulk solution. In this section we therefore regard the 5 bulk parameters λ_1 , λ_2 , ξ_1 , q and ω as given (we remove both p and ξ_2 using appropriate coordinate conditions), and look for solutions for the kinds of branes which can smooth the singularities at $\eta = \pm\infty$.

We emphasize that our purpose here is simply to show that a regularization procedure exists for any choice of bulk solution, through an appropriate choice for the 4-branes and caps. We return in subsequent sections to the relations which must exist between the parameters governing the branes and caps, due to the interpolation between them of a 4D flat bulk.

Parameter counting. It is instructive to count parameters and constraints, to get a sense of whether or not the problem of capping a given bulk is over-determined. To this end it is worth distinguishing between those parameters which are integration constants in the capped region, and those which arise within the action, \mathcal{S} , governing the 4-brane. We start by counting only those relations which are independent of the 4-brane action, before returning to those which are not.

\mathcal{S} -independent conditions: we have seen that each cap naively involves 7 integration constants, λ_a , ξ_{1a} , ξ_{2a} , p_a , q_a , ω_a and η_a that are related by the smooth-geometry condition, (2.9), at each cap. Counting the two caps this gives a total of $6 + 6 = 12$ independent cap integration constants.

At each cap these parameters are subject to 3 continuity conditions, eqs. (3.3)–(3.5), as well as the 1 jump condition, (3.24) or (3.25), constructed by eliminating $U(\phi)$ and $V(\phi)$ from eqs. (3.21)–(3.23). The topological constraint then imposes one more overall relation which relates the properties of the bulk to those of both caps, giving a grand total of $4 + 4 + 1 = 9$ conditions. Barring other obstructions we then expect to find a $12 - 9 = 3$ -parameter family of capped geometries which can match properly to the given bulk.

\mathcal{S} -dependent conditions: in addition to these are the parameters $U_a(\phi)$ and $V_a(\phi)$ governing the 4-brane action, \mathcal{S} . For each brane these two functions are related by the three remaining conditions, eqs. (3.21), (3.22) and (3.27). Solving the two linear equations, (3.21) and (3.22), immediately gives U_a and V_a as explicit functions of η_a : $U_a = U_a(\eta_a)$ and $V_a = V_a(\eta_a)$ (where we suppress the dependence on the other cap and bulk parameters).

We are then left with one remaining relation: the dilaton jump condition, eq. (3.27). Since this requires knowing the derivatives, $dU_a/d\phi$ and $dV_a/d\phi$, further progress requires making some choices for the functional form of these quantities.

- If U_a and V_a are both constant, then both are fixed by eqs. (3.21) and (3.22). In this case the dilaton jump condition, eq. (3.27), imposes an additional 2 constraints (one at each cap) on the 3 cap integration constants which remain to this point undetermined. We are then led to expect a 1-parameter family of capped solutions.
- If U_a and V_a preserve scale invariance, then $U_a = u_a e^{-\phi/2}$ and $V_a = v_a e^{\phi/2}$, have 2 free parameters. In this case the counting naively goes through as above, with one change: although u_a and v_a are fixed by solving the Israel junction conditions, eqs. (3.21) and (3.22), the dilaton jump condition, eq. (3.27), degenerates to $\lambda_3 = 0$ and so does not further constrain any 4-brane or cap parameters. (None of these matching conditions fix the scale symmetry $\omega \rightarrow \omega + \Delta$, $\omega_a \rightarrow \omega_a + \Delta$ and $\omega_b \rightarrow \omega_b + \Delta$. However, because we here regard the bulk parameter ω to have been specified this symmetry does not preclude the determination of ω_a and ω_b in terms of ω .) We are therefore led in this case to 3 free parameters in the capped solution.
- More general choices for U_a and V_a potentially involve more parameters, and so allow more freedom of choice for the capped geometry. For instance, if $U_a = u_a e^{s_a \phi}$ and $V_a = v_a e^{t_a \phi}$, then the three conditions, (3.21), (3.22) and (3.26), provide three relations amongst the four parameters u_a , v_a , s_a and t_a , and in particular (3.26) no longer constrains the parameters of the caps. In this case we'd expect a total of 5 free parameters to describe the capped geometry.

Considerations such as these lead us to expect that capped solutions of the type we entertain should exist for any given kind of bulk geometry, barring an obstruction to solving the relevant equations. Furthermore, we expect to find at least a 1-parameter family of such solutions, and this has a simple physical interpretation: in the absence of the topological constraint the caps have 2 free parameters, corresponding to the freedom to choose the positions, η_a and η_b , where we choose to position the two caps. The topological constraint can then impose one relation amongst these two positions, relating them to the quantum number, N , which governs the total amount of Maxwell flux.

Notice that our counting here regards U and V as parameters to be adjusted even though these arise within the brane action rather than as integration constants in the solutions to the field equations. So the existence of the caps requires these parameters in the action to be tuned relative to one in a way which depends on the properties of the given bulk solution. We also do not distinguish here whether the solutions found give positive

values for U and V , as would normally be required by positivity of the kinetic energy associated with brane motion (V) and the Stueckelberg field (U).

Freely-floating 4-branes. The previous section takes the point of view that the ϕ -dependence of the 4-brane action can be arbitrarily parameterized, with the parameters required to cap the given bulk geometry being fixed in terms of the positions of the caps and other variables. Another point of view is to ask for a 4-brane action to be defined so that the same 4-brane action can be used at *any* 4-brane position, for a given bulk geometry. As we shall see, consistency also requires the cap geometry to be varied as a function of the brane position. This approach is similar in spirit to what is done for the actions of end-of-the-world branes which mark the boundary of bulk spaces in discussions of the AdS/CFT correspondence [38].

This amounts to asking that the η_a -dependence inferred by solving eqs. (3.21) and (3.22) for $U_a(\eta_a)$ and $V_a(\eta_a)$ is completely given by the implicit η_a -dependence which U_a and V_a inherit as functions of $\phi(\eta_a)$ (with η_a -independent constants). That is, we demand $U_a(\eta_a) = U_a[\phi(\eta_a)]$ and $V_a(\eta_a) = V_a[\phi(\eta_a)]$. We call such a 4-brane action the ‘floating’ action which is defined by the given bulk and capped geometries. In principle, the functional form that this requires for both $U_a(\phi)$ and $V_a(\phi)$ can be inferred in this way using the known expressions for the bulk dilaton profile, $\phi(\eta_a)$, together with the expressions for $U_a(\eta_a)$ and $V_a(\eta_a)$ obtained by solving eqs. (3.21) and (3.22).

Finally, the dilaton jump condition, (3.27), is then read as an additional constraint on the parameters which govern the capped geometry. To identify this constraint more explicitly, we notice that we could use either the bulk dilaton profile, $\phi(\eta)$, or the profile in the cap, $\hat{\phi}(\eta)$, to convert the η_a dependence of U_a and V_a into their dependence on the dilaton. In particular, we have two ways of evaluating the dilaton derivative of the 4-brane quantities like U_a , which must agree with each other:

$$\left(\frac{dU_a}{d\eta_a}\right) = \left(\frac{dU_a}{d\phi}\right)_{\phi=\phi(\eta)} \left(\frac{d\phi}{d\eta_a}\right) = \left(\frac{dU_a}{d\phi}\right)_{\phi=\hat{\phi}(\eta)} \left(\frac{d\hat{\phi}}{d\eta_a}\right). \quad (4.1)$$

Here $d\phi/d\eta_a = (\partial\phi/\partial\eta)|_{\eta\rightarrow\eta_a}$, while $d\hat{\phi}/d\eta_a$ also includes the implicit dependence on η_a that that $\hat{\phi}$ acquires through its dependence on the η_a -dependent cap parameters. Collectively denoting these cap parameters by $\{\hat{c}_s\} = \{\lambda_a, \xi_{1a}, \dots\}$, we have

$$\frac{d\hat{\phi}}{d\eta} = \left[\left(\frac{\partial\hat{\phi}}{\partial\eta}\right) + \left(\frac{\partial\hat{\phi}}{\partial\hat{c}_s}\right) \frac{\partial\hat{c}_s}{\partial\eta_a} \right]_{\eta\rightarrow\eta_a}. \quad (4.2)$$

The desired consistency condition on the cap parameters comes from equating $(\partial\hat{\phi}/\partial\eta)_{\eta\rightarrow\eta_a}$ obtained by solving eqs. (4.1) and (4.2), with that inferred from the dilaton jump condition, eq. (3.27).

We see from this that the number of independent constraints on the cap geometry is the same as it was when we made the simpler assumption that U and V were constants. We have not yet tried to solve these constraints to determine the functional form for $U_a(\phi)$ and $V_a(\phi)$ which would be obtained.

Solving the matching conditions. In order to see in more detail if obstructions to solutions to the matching conditions might exist, we next examine some of these conditions in more detail. Recall the counting: each cap is described by 7 integration constants: λ_a , q_a , ξ_{1a} , ξ_{2a} , ω_a , p_a and η_a , if the smoothness condition is not used, for a total of 14 once both branes are included. Smoothness of the caps and continuity at both branes, with the topological condition cut these down by a total of 9 conditions, leaving 5 undetermined. There is also one combination of jump conditions at each brane which does not involve the potentials U and V , reducing us to 3 parameters. If U and V are ϕ -independent, then the dilaton jump condition for each brane removes 2 more. This leaves 1 cap parameter undetermined. By contrast, the integers k_a , k_b and N describing the monopole flux and background configuration for the Stueckelberg field are not solved for, but are instead regarded as choices we get to pick by hand. We show there is a solution to the junction conditions for a range of k_a , k_b and N .

A special case: before examining the general case, we first examine in detail a special case where all of the conditions may be explicitly solved in closed form. In order to do this, we make the following *ansatz* for the integration constants, λ_a and λ_b :

$$\frac{\lambda_a}{q_a} = \frac{\lambda_1}{q} = \frac{\lambda_b}{q_b}, \tag{4.3}$$

Then, we choose ω_a and ω_b to satisfy

$$\omega - \omega_a + \lambda_3 \eta_a = 0 = \omega - \omega_b + \lambda_3 \eta_b \tag{4.4}$$

while the parameters q_a and q_b are chosen such that

$$\frac{q_a}{g_a} = \frac{q \lambda_2}{g \lambda_1} = \frac{q_b}{g_b} \tag{4.5}$$

The motivation for these choices comes from the way they simplify the continuity equations.

Eq. (4.3) ensures that the continuity relation, eq. (3.3), simplifies to

$$\lambda_a (\eta_a - \xi_{1a}) = \lambda_1 (\eta_a - \xi_1), \tag{4.6}$$

which we solve for ξ_{1a} , giving

$$\xi_{1a} = \eta_a - \frac{\lambda_1}{\lambda_a} (\eta_a - \xi_1). \tag{4.7}$$

Similarly, eqs. (4.4) and (4.5) allow the continuity relation (3.4) to be written

$$\lambda_a (\eta_a - \xi_{2a}) = \lambda_2 (\eta_a - \xi_2), \tag{4.8}$$

with solution

$$\xi_{2a} = \eta_a - \frac{\lambda_2}{\lambda_a} (\eta_a - \xi_2). \tag{4.9}$$

Similar results follow for ξ_{1b} and ξ_{2b} using identical arguments.

Given these conditions, the topological constraint, (3.9), degenerates into

$$\frac{N}{e} = \frac{2\lambda_1}{q}, \quad (4.10)$$

which is independent of the brane positions, and so can be regarded as a condition on the background field gauge coupling, e (which can be altered by adjusting how the background gauge field is embedded into the gauge group). Similarly, using the choices (4.4) and (4.5) in (3.24), derived from the jump conditions, leads to the considerably simpler form

$$\frac{2k_a}{N} = 1 + \frac{(\lambda_3/\lambda_1)}{2[1 - (q_a/q)]}, \quad (4.11)$$

with a similar result for brane b . For $\lambda_3 = 0$ this last formula requires N to be even, and was obtained previously for non-supersymmetric 6D models in ref. [25]. If $\lambda_3 \neq 0$, on the other hand, it instead can be read as giving q_a/q in terms of λ_3 . Identical considerations similarly apply to brane b . Due to the condition (4.5), the condition (4.11) allows to obtain a constraint that the parameter g_a must satisfy in order to get a solution:

$$\frac{2k_a}{N} = 1 + \frac{(\lambda_3/\lambda_1)}{2[1 - g_a\lambda_2/(g\lambda_1)]}. \quad (4.12)$$

Next, given assumption (4.3), the smoothness condition, eq. (2.9), reduces to

$$2g_a e^{\lambda_a(\xi_{2a} - \xi_{1a})} = \frac{q}{\lambda_1} = 2g_b e^{\lambda_b(\xi_{1b} - \xi_{2b})}, \quad (4.13)$$

which, using eqs. (4.7) and (4.9), can be reformulated as

$$e^{(\lambda_1 - \lambda_2)\eta_a + \lambda_2\xi_2 - \lambda_1\xi_1} = \frac{q}{2\lambda_1 g_a}. \quad (4.14)$$

This may be regarded as the condition that determines the brane position η_a . Notice that this last expression, together with its counterpart for brane b , gives the following constraint relating the positions of the two branes:

$$(\lambda_1 - \lambda_2)(\eta_a - \eta_b) = \ln\left(\frac{q^2}{4\lambda_1^2 g_a g_b}\right). \quad (4.15)$$

The final parameter, p_a , is fixed by eq. (3.5) to be $p_a = \lambda_3\eta_a$.

Finally, we solve the dilaton jump condition and the two Israel junction conditions, which involve the 4-brane parameters U , V , $dU/d\phi$ and $dV/d\phi$. Solving the two Israel conditions gives the following expressions for U_a and V_a :

$$\begin{aligned} -2\mathcal{W}^4 \mathcal{A} V_a &= \frac{\lambda_3}{2} + 2(\lambda_a - \lambda_2) \tanh\lambda_2 (\eta_a - \xi_2) + (\lambda_1 - \lambda_a) \tanh\lambda_1 (\eta_a - \xi_1) \\ -\frac{\mathcal{W}^4}{\mathcal{A}} (k_a - eA_\psi)^2 U_a &= \frac{\lambda_3}{2} + (\lambda_a - \lambda_1) \tanh\lambda_1 (\eta_a - \xi_1). \end{aligned} \quad (4.16)$$

The dilaton matching condition similarly becomes

$$2\lambda_3 - (\lambda_a - \lambda_1) \tanh\lambda_1 (\eta_a - \xi_1) + (\lambda_a - \lambda_2) \tanh\lambda_2 (\eta_a - \xi_2) = \mathcal{F} \left(\frac{dU_a}{d\phi}, \frac{dV_a}{d\phi} \right) \quad (4.17)$$

where the function \mathcal{F} denotes the combination of the U and V and their derivatives appearing on the right-hand-side of (3.26) (and so $\mathcal{F} = 0$, in particular, if $dU_a/d\phi = dV_a/d\phi = 0$).

As usual, whether this last equation must be read as a new constraint depends on the functional form which is assumed for $U_a(\phi)$ and $V_a(\phi)$. In particular, if U_a and V_a are constants (or scale invariant), then eq. (4.17) imposes non-trivial additional conditions on the parameters of the cap geometries, and so generically can obstruct the existence of a cap geometry unless the bulk parameters are tuned to assure its satisfaction.

Notice that the necessity to tune parameters in the bulk and cap actions arises in this case because the initial simplifying *ansätze*, eqs. (4.3), (4.4) and (4.5), make the matching problem into an over-determined problem, rather than allowing the 1-parameter family of solutions which are possible in the generic case.

The general case: we now return to solving the matching condition in the general case, not subject to the *ansätze*, eqs. (4.3), (4.4) and (4.5). It is convenient to define first the quantities

$$\Lambda_{ia} = \lambda_a(\eta_a - \xi_{ia}), \quad \Lambda_{ib} = \lambda_b(\eta_b - \xi_{ib}), \quad (4.18)$$

$$\text{and} \quad \Delta_{ia} = \lambda_i(\eta_a - \xi_i), \quad \Delta_{ib} = \lambda_i(\eta_b - \xi_i) \quad (4.19)$$

where $i = 1, 2$. In our counting, the parameters Λ_{ia} and Λ_{ib} replace ξ_{ia} and ξ_{ib} , whereas Δ_{ia} and Δ_{ib} are known functions of η_a and η_b .

Recall that there are a total of 14 cap parameters, and these are subject to a total of 11 conditions before the three conditions (per brane) involving U and V are used, leaving 3 parameters undetermined. (Depending on what we assume about the 4-brane action — such as if U and V are constants — two of these can then be fixed by the dilaton jump conditions, leaving the single undetermined parameter, although we do not yet apply this constraint in this section.) Although other choices are possible, we find it easiest to solve for the cap parameters as functions of the three undetermined quantities $(\eta_a, \eta_b, \Lambda_{1b})$.

We start with the topological constraint, eq. (3.9), which we simplify by using eq. (3.3) and its counterpart for brane b to eliminate the combinations λ_a/q_a and λ_b/q_b . Using the resulting expressions in eq. (3.9) gives

$$\tanh\Delta_{1b} - \tanh\Delta_{1a} = \frac{qN}{e\lambda_1} - \frac{\varepsilon_a e^{\Lambda_{1a}}}{\cosh\Delta_{1a}} - \frac{\varepsilon_b e^{-\Lambda_{1b}}}{\cosh\Delta_{1b}}, \quad (4.20)$$

where we define $\varepsilon_a = |q_a|/q_a = \text{sign } q_a$, and similarly for ε_b and ε . Writing this as $\varepsilon_a e^{\Lambda_{1a}} = F$, with $F = F(\eta_a, \eta_b, \Lambda_{1b})$ given by

$$F = \cosh\Delta_{1a} \left(\frac{qN}{e\lambda_1} - \frac{\varepsilon_b e^{-\Lambda_{1b}}}{\cosh\Delta_{1b}} - \tanh\Delta_{1b} + \tanh\Delta_{1a} \right), \quad (4.21)$$

shows that solutions exist so long as we choose $\varepsilon_a = \text{sign } F$, and gives these solutions as

$$\Lambda_{1a} = \ln |F|. \quad (4.22)$$

Using the smoothness condition together with the continuity condition, eq. (3.3), and the above solution for Λ_{1a} , then gives

$$\Lambda_{2a} = \ln \left| \frac{\lambda_1 g_a (1 + F^2)}{q \cosh \Delta_{1a}} \right|. \quad (4.23)$$

As we have now solved for Λ_{1a} and Λ_{2a} in terms of η_a , η_b , and Λ_{1b} , we do not bother to eliminate these two parameters from future expressions.

We next solve for λ_a . Starting from eq. (3.24) and using the continuity conditions to simplify further, we arrive at the expression

$$\lambda_a = \frac{1}{\tanh \Lambda_{1a}} \left(\lambda_1 \tanh \Delta_{1a} - \frac{\lambda_3}{2} + \left[1 - \frac{\varepsilon \varepsilon_a g_a \lambda_2 \cosh \Delta_{1a} \cosh \Lambda_{2a}}{g \lambda_1 \cosh \Delta_{2a} \cosh \Lambda_{1a}} \right] \left[\frac{q k_a}{e} - \frac{\varepsilon_a \lambda_1 e^{\Lambda_{1a}}}{\cosh \Delta_{1a}} \right] \right). \quad (4.24)$$

It is important to note that by choosing the integer k_a appropriately, we can ensure $\lambda_a > 0$.⁵ Again, as we have solved for λ_a in terms of the three required parameters, we will not need to eliminate it from future equations. Finally, the 3 continuity equations at brane a directly give

$$p_a = \lambda_3 \eta_a, \quad (4.25)$$

$$q_a = \left(\frac{\varepsilon q \lambda_a}{\lambda_1} \right) \left(\frac{2F}{1 + F^2} \right) \cosh \Delta_{1a}, \quad (4.26)$$

$$\omega_a = \omega + \lambda_3 \eta_a + \frac{1}{2} \ln \left| \frac{g_a \lambda_2 \cosh \Lambda_{2a}}{g \lambda_a \cosh \Delta_{2a}} \right|. \quad (4.27)$$

The analysis at brane b is similar, for which we find

$$\Lambda_{2b} = \Lambda_{1b} + \ln \left| \frac{q \cosh \Delta_{1b}}{2 \lambda_1 g_b \cosh \Lambda_{1b}} \right|, \quad (4.28)$$

$$\lambda_b = \frac{1}{\tanh \Lambda_{1b}} \left(\lambda_1 \tanh \Delta_{1b} - \frac{\lambda_3}{2} + \left[1 - \frac{\varepsilon \varepsilon_b g_b \lambda_2 \cosh \Delta_{1b} \cosh \Lambda_{2b}}{g \lambda_1 \cosh \Delta_{2b} \cosh \Lambda_{1b}} \right] \left[\frac{q k_b}{e} + \frac{\varepsilon_b \lambda_1 e^{-\Lambda_{1b}}}{\cosh \Delta_{1b}} \right] \right), \quad (4.29)$$

and

$$p_b = \lambda_3 \eta_b, \quad (4.30)$$

$$|q_b| = \left(\frac{|q| \lambda_b}{\lambda_1} \right) \frac{\cosh \Delta_{1b}}{\cosh \Lambda_{1b}}, \quad (4.31)$$

$$\omega_b = \omega + \lambda_3 \eta_b + \frac{1}{2} \ln \left| \frac{g_b \lambda_2 \cosh \Lambda_{2b}}{g \lambda_b \cosh \Delta_{2b}} \right|. \quad (4.32)$$

⁵One might worry that this is no longer true if the first term in square brackets is zero, but a little work shows that the condition for this term being nonzero (for arbitrary k_a) is equivalent to the condition $U_a \neq 0$, which we assume.

By using the previous expressions for Λ_{2b} and λ_b , we see that we have solved for $|q_b|$ and ω_b in terms of the required 3 parameters. The sign of q_b can be determined by the gauge field jump condition at brane b .

This exhausts all of the matching conditions which do not involve the 4-brane coupling functions. The value of these functions, U and V , at each brane is then easily obtained by solving the Israel junction conditions, eqs. (3.21) and (3.22), leaving only the dilaton jump condition to be solved. If U and V contain enough parameters to allow them and their derivatives to be varied independently for each brane, then this last condition can be solved without adding further constraints on the parameters of the cap geometry.

Alternatively, when $dU/d\phi$ and $dV/d\phi$ are not independent of U and V — such as when U and V are both ϕ -independent, or are scale invariant — then the dilaton matching condition, eq. (3.26), imposes an additional constraint. After some manipulation this can be written in the form

$$g \cosh\Delta_{2a} [2\lambda_3 + \lambda_1 \tanh\Delta_{1a} - \lambda_2 \tanh\Delta_{2a}] = \frac{g_a^2 \lambda_1 \lambda_2}{2q \cosh\Delta_{1a}} \left\{ \left(\frac{q^2}{\lambda_1^2 g_a^2} \right) \cosh^4\Delta_{1a} \left[\frac{(\tilde{N} - \varepsilon_b e^{-\Lambda_{1b} \text{sech}\Delta_{1b}})^2}{1 + (\tilde{N} - \varepsilon_b e^{-\Lambda_{1b} \text{sech}\Delta_{1b}})^2} \right] - 1 \right\} \quad (4.33)$$

where we define the quantity

$$\tilde{N} = \frac{qN}{e\lambda_1} - \tanh\Delta_{1b} + \tanh\Delta_{2a}. \quad (4.34)$$

A particularly useful special of this condition takes η_a to be very large and negative (and η_b to be large and positive). This is a limit of particular interest because it corresponds to the cap volume being much smaller than that of the bulk (more about this in subsequent sections). In this limit we have $\tilde{N} \approx -2 + qN/(e\lambda_1)$ and the previous equation reduces to

$$g \cosh\Delta_{2a} [2\lambda_3 - \lambda_1 + \lambda_2] = \left(\frac{\lambda_2 q}{2\lambda_1} \right) \cosh^3\Delta_{1a} \left[\frac{(\tilde{N} - \varepsilon_b e^{-\Lambda_{1b} \text{sech}\Delta_{1b}})^2}{1 + (\tilde{N} - \varepsilon_b e^{-\Lambda_{1b} \text{sech}\Delta_{1b}})^2} \right]. \quad (4.35)$$

Recall that eq. (4.33) — or eq. (4.35) — and its counterpart for cap ‘ b ’ impose two conditions on the three remaining free cap parameters, η_a , η_b and Λ_{1b} . In particular, in the limit of large negative η_a and large positive η_b , this equation is easily solved for η_a because \tilde{N} is independent of η_a and η_b . In general, the freedom to choose N can be used to help ensure that solutions exist.

4.2 Bulk geometries sourced by given branes

In the previous section the bulk geometry is considered to be given, and we ask whether regularizing caps can be constructed. This section adopts a different point of view, wherein the characteristics of the caps — i.e. all the integration constants that define the cap geometry and the quantities U and V — are given, and we seek the properties of the bulk

which results. In particular, our interest is to see whether and how the two caps must be related to one another, and to check whether the bulk configuration is always of the form of a GGP solution, with flat four dimensional slices.

Our goal in doing so is two-fold. First, in this subsection, we wish to see whether this reduced problem is over-determined, and if so what is required in detail of the branes in order to ensure a solution. Secondly, in section 4.3 we set out to understand how the volume of the bulk geometry is related to the brane properties, and, by doing so, to exhibit a stabilization mechanism for the bulk volume. Of particular interest is then to understand what 4-brane/cap properties are required to ensure the volumes of the capped regions are much smaller than that of the intervening bulk (as is required if the 4-branes and caps describe the microscopic structure of more macroscopic 3-branes).

Parameter counting and junction conditions. We now show that counting equations and parameters suggests we are not completely free to specify the 4-brane action for brane a arbitrarily if we ask that it interpolate between 4D flat cap and bulk geometries. This can be done only if the 4-brane action is subject to one constraint equation (as was argued in ref. [35]), but once this is satisfied there is sufficient information to determine the parameters describing both the bulk geometry *and* the properties of brane b .⁶

To this end, imagine we specify the cap geometry and 4-brane action at a given position $\eta = \eta_a$. Next recall that there are 7 integration constants characterizing the the bulk geometry — $\lambda_1, \lambda_2, \xi_1, \xi_2, p, q$ and ω . (Notice that, although previously we have removed two of these quantities — ξ_1 and p — by suitably adjusting coordinates, this is typically no longer possible without altering the specified parameters for cap a .) These 7 parameters are subject to a total of 7 conditions at η_a , consisting of 3 continuity conditions (metric and dilaton) and 4 jump conditions (Israel, Maxwell and dilaton), suggesting that the bulk parameters are completely specified in terms of those of the cap and 4-brane at η_a .

As we show in the next section, however, one of these seven equations which is *supposed* to determine one of the bulk parameters turns into a constraint equation amongst cap and brane parameters. Thus, what we find is that for any given cap and brane which satisfies the constraint, there is a one-parameter family of flat bulks to which we can match. Physically, it is easiest to interpret this one parameter in the coordinate system where $\xi_1 = \xi_{1a} = 0$. Recall that in this coordinate system the brane location in the bulk and cap is η_a and $\hat{\eta}_a$, respectively, where these two numbers are generically not the same. Here, we again imagine fixing the cap and brane properties at $\hat{\eta}_a$, and then solving for six of the seven bulk parameters: $\lambda_1, \lambda_2, \xi_2, \omega, p, q$, and η_a . Thus, this one-parameter family of bulk solutions corresponds to where we choose to place the brane in the bulk coordinate system. If η_a is fixed, then we find a unique solution for the bulk.

Continuing to use the coordinate system where $\xi_1 = \xi_{1a} = \xi_{1b} = 0$, we see that once the bulk geometry is thus inferred, there remain 10 parameters associated with cap b , consisting of 6 integration constants — $\lambda_b, q_b, \xi_{2b}, \omega_b, p_b$, and $\hat{\eta}_b$ — plus the brane position

⁶To be precise, we find a two parameter family of solutions for the bulk and cap b , corresponding to where we choose to embed the two branes in the bulk. Once this choice is made, then the bulk and cap b are unique.

η_b in the bulk coordinate system, the two 4-brane parameters, U_b and V_b , and *one* linear combination of their derivatives. These 10 parameters are then subject to 9 conditions, consisting of the 7 continuity and jump conditions at the brane location, the smoothness condition at $\eta \rightarrow \infty$ for cap b and the topological constraint on the Maxwell field. Provided there are no obstructions to solving these equations, this shows that once we choose the properties of one brane (subject only to the Hamiltonian constraint), together with the location of the two branes in the bulk coordinate system, η_a and η_b , then properties of the other brane and the intervening 4D-flat bulk are precisely dictated. If the properties of brane b are not adjusted in this way in terms of those of brane a then the intervening bulk solution cannot be 4D flat, and instead must either be 4D maximally symmetric but not flat [30] or time-dependent and not Lorentz invariant [33].

This counting bears out, and make more precise, expectations based on earlier studies of the general properties of bulk solutions to 6D supergravity. In particular, for 4D maximally-symmetric solutions [30] (including those which are not 4D flat) the bulk geometry depends nontrivially only on η , and so we may imagine integrating the bulk field equations in the η direction, starting at brane a and ending at brane b . Since the η - η Einstein equation does not involve second derivatives of the metric, it represents a ‘Hamiltonian’ constraint on those ‘initial’ conditions at brane a which can be consistently used for such an integration. In this language, the above-mentioned constraint on the allowed 4-brane parameters corresponds to requirements imposed on the 4-brane by matching to the Hamiltonian constraint in the bulk, restricted to 4D flat geometries [35]. Furthermore, since the bulk geometry is completely specified by integrating forward in η using the ‘initial’ conditions at brane a , its asymptotic form at brane b is seen to be completely determined, in agreement with what we find here for explicit 4-brane/cap regularizations of this asymptotic form.

Explicit solutions. To better see if parameter and equation counting provides the whole story, we next solve the matching to see whether obstructions to their solutions can exist.

The bulk. The continuity equations, eqs. (3.3) – (3.5), read in this case:

$$|q| = |q_a| \left(\frac{\lambda_1 \cosh \Lambda_{1a}}{\lambda_a \cosh \Delta_{1a}} \right), \tag{4.36}$$

$$e^{-2\omega} = e^{-2(\omega_a - \lambda_3 \eta_a)} \left(\frac{g_a \lambda_2 \cosh \Lambda_{2a}}{g \lambda_a \cosh \Delta_{2a}} \right), \tag{4.37}$$

$$p = p_a - \lambda_3 \eta_a, \tag{4.38}$$

and can be thought as fixing the bulk parameters q , ω , and p (recall the definitions of the parameters Λ and Δ in formulae (4.18) and (4.19)). Note that the sign of q is not yet fixed. These solutions are given in terms of the four bulk quantities λ_1 , λ_2 , Δ_{1a} , and Δ_{2a} , for which we now solve.

Before proceeding it is convenient to first define four combinations of brane and cap parameters:

$$C_1 = \left(\frac{g_a \cosh \Lambda_{2a}}{2g \lambda_a} \right) \left[\hat{\mathcal{A}} \hat{\mathcal{W}}^4 (V_a - 2V'_a) - \frac{\hat{\mathcal{W}}^4}{2\hat{\mathcal{A}}} (U_a + 2U'_a) (k_a - e \hat{A}_\psi)^2 \right], \tag{4.39}$$

$$\mathcal{C}_2 = \left(\frac{g_a \cosh \Lambda_{2a}}{4g\lambda_a} \right) \left[\hat{\mathcal{A}} \hat{\mathcal{W}}^4 (5V_a - 2V'_a) + \frac{\hat{\mathcal{W}}^4}{2\hat{\mathcal{A}}} (3U_a - 2U'_a) (k_a - e\hat{A}_\psi)^2 + 4\lambda_a \tanh \Lambda_{2a} \right], \quad (4.40)$$

$$\mathcal{C}_3 = \left(\frac{\varepsilon_a g_a \cosh \Lambda_{2a}}{g \cosh \Lambda_{1a}} \right) \left[-\frac{eU_a}{q_a} \left(\frac{\hat{\mathcal{W}}^4}{\hat{\mathcal{A}}} \right) (k_a - e\hat{A}_\psi) + 1 \right], \quad (4.41)$$

$$\mathcal{C}_4 = \left(\frac{g_a \cosh \Lambda_{2a}}{4g\lambda_a} \right) \left[\hat{\mathcal{A}} \hat{\mathcal{W}}^4 (V_a - 2V'_a) + \frac{\hat{\mathcal{W}}^4}{2\hat{\mathcal{A}}} (7U_a - 2U'_a) (k_a - e\hat{A}_\psi)^2 + 4\lambda_a \tanh \Lambda_{1a} \right], \quad (4.42)$$

where primes here denote differentiation with respect to ϕ . These four parameters will take the place of $U_a(\eta_a)$, $V_a(\eta_a)$, their derivatives (which appear in only one linear combination), and k_a . The action for brane a can therefore be equally well characterized by these four quantities, as by our original parameterization in terms of $U_a(\eta_a)$, $V_a(\eta_a)$, and derivatives. With these definitions in hand, the remaining four matching conditions reduce to the following equations:

$$\mathcal{C}_1 = \cosh \Delta_{2a} \left[1 - \left(\frac{\lambda_1}{\lambda_2} \right) \right]^{\frac{1}{2}} \quad (4.43)$$

$$\mathcal{C}_2 = \sinh \Delta_{2a} \quad (4.44)$$

$$\mathcal{C}_3 = \varepsilon \left(\frac{\lambda_1}{\lambda_2} \right) \frac{\cosh \Delta_{2a}}{\cosh \Delta_{1a}} \quad (4.45)$$

$$\mathcal{C}_4 = \left(\frac{\lambda_1}{\lambda_2} \right) \tanh \Delta_{1a} \cosh \Delta_{2a}. \quad (4.46)$$

Recalling that both λ_1 and λ_2 are positive, we see immediately that $\varepsilon \equiv \text{sign } q = \text{sign } \mathcal{C}_3$.

We note that this system of equations is over-determined, since there are four equations but only three unknowns: Δ_{1a} , Δ_{2a} , and λ_1/λ_2 . In fact, by squaring the above equations it is straightforward to check this constraint is given by

$$\mathcal{C}_1^2 - \mathcal{C}_2^2 + \mathcal{C}_3^2 + \mathcal{C}_4^2 = 1. \quad (4.47)$$

When the above equation is satisfied, then it can be shown that the bulk fields satisfy the Hamiltonian constraint which ensures 4D flatness.⁷ Henceforth, we assume that the brane properties are chosen such that the Hamiltonian constraint is satisfied. In this case, the solution to eqs. (4.43)–(4.46) is

$$\Delta_{1a} = \text{sign}(\mathcal{C}_4) \text{arcosh} \left[\left(1 + \frac{\mathcal{C}_4^2}{\mathcal{C}_3^2} \right)^{\frac{1}{2}} \right], \quad (4.48)$$

$$\Delta_{2a} = \text{arsinh}(\mathcal{C}_2), \quad (4.49)$$

$$\frac{\lambda_1}{\lambda_2} = \left(1 - \frac{\mathcal{C}_1^2}{\mathcal{C}_1^2 + \mathcal{C}_3^2 + \mathcal{C}_4^2} \right)^{\frac{1}{2}}. \quad (4.50)$$

⁷This Hamiltonian constraint is given by eq. (34) in reference [30].

where the range of arcosh is taken to be $\{x \in \mathbb{R} : x \geq 0\}$. It is easy to see that solutions to these equations exist for any values of the \mathcal{C}_i , subject only to the constraint that they obey eq. (4.47).

As expected from the arguments in the previous section, we indeed find a one-parameter family of possible bulks. Once this parameter is fixed — corresponding to choosing where in the bulk we wish to embed the brane — then the bulk solution becomes unique. Henceforth, we assume that this choice has been made (as can be accomplished by making a specific choice for p in eq. (4.38)).

Cap b . Having uniquely determined the bulk solution, it remains to determine the properties of the 4-brane and cap at brane b . In order to find a unique solution, we first specify the location where we wish to cap the bulk, η_b . Since this analysis is identical to that of section 4.1, we do not repeat it in detail here, however the three continuity conditions, the smoothness condition, and the combination of the jump conditions which is independent of U_b and V_b provide 5 constraints on the 5 cap integration constants p_b , Λ_{2b} , λ_b , q_b and ω_b (see eqs. (4.28) - (4.32)). Then, the two Israel junction conditions fix U_b and V_b , and the dilaton jump condition provides the constraint which fixes the one relevant combination of derivatives U'_b and V'_b . The only cap parameter which is not fixed by these conditions is Λ_{1b} , and this can be determined from the topological equation (4.20). As expected [30], both the properties of the bulk and those of the 4-brane and cap at $\eta = \eta_b$ are dictated by those of the brane and cap at $\eta = \eta_a$.

4.3 Volume stabilization and large hierarchy

The previous analysis fixing the seven bulk integration constants in terms of given cap parameters fixes in particular the integration constant, ω , that parameterizes the bulk volume. This provides a natural 6D mechanism for stabilizing this bulk volume. In this section we explore this stabilization in more detail, focussing on the conditions which are required to obtain a large hierarchy between the volumes of the bulk and the caps. In the next section we identify the low-energy 4D effective potential which is generated in this way for ω .

Conditions for a hierarchy. We now ask for the conditions the brane and cap actions should satisfy to ensure that the cap volumes are much smaller than those of the bulk. Our point of view here is that the bulk geometry is given, and so would like to phrase the conditions for a hierarchy in terms of only those parameters over which we have control: the bulk parameters and the three cap parameters, η_a , η_b , and Λ_{1b} .

In order to have branes whose circumference is small, we seek to ensure $\mathcal{A}(\eta_a)$ and $\mathcal{A}(\eta_b)$ are much less than one. We see from eq. (2.5) that it is natural to examine for this purpose the limit $\eta_a \rightarrow -\infty$ and $\eta_b \rightarrow \infty$, although in general this need not be sufficient in itself to have small cap volumes. However, we now argue that sufficient conditions for obtaining small cap volumes are given by

$$\begin{aligned} \Lambda_{1a} &= \lambda_a(\eta_a - \xi_{1a}) \ll -1 \\ \Lambda_{2a} &= \lambda_a(\eta_a - \xi_{2a}) \ll -1 \end{aligned} \tag{4.51}$$

with similar conditions for brane b . Large, negative η_a is not sufficient for small cap volumes because it does not in itself ensure that these conditions are satisfied. Under these conditions we may use the asymptotic form for the hyperbolic functions and so obtain the following expression for volume of cap a

$$\begin{aligned} \Omega_a &= 2\pi \int_{-\infty}^{\eta_a} d\eta \hat{\mathcal{A}}^2 \hat{\mathcal{W}}^4 \\ &\simeq \frac{\pi}{\lambda_a} e^{2(\omega - \omega_a + p_a)} (\mathcal{A}^2 \mathcal{W}^4)|_{\eta_a} \end{aligned} \tag{4.52}$$

In arriving at the second line we have used the continuity equations, (3.1), to relate cap functions to bulk functions. The cap volume must be compared with the bulk volume, given by the expression

$$\begin{aligned} \Omega_{bulk} &= 2\pi \int_{\eta_a}^{\eta_b} d\eta A^2 W^4 \\ &= \left(\frac{(2\pi)^2 \lambda_1 \lambda_2^3 e^{2\omega}}{(2g)^3 q} \right)^{\frac{1}{2}} \int_{\eta_a}^{\eta_b} d\eta \frac{e^{\lambda_3 \eta}}{\cosh^{\frac{3}{2}}[\lambda_2(\eta - \xi_2)] \cosh^{\frac{1}{2}}[\lambda_1(\eta - \xi_1)]}. \end{aligned} \tag{4.53}$$

It is simple to check that the integral in the previous expression is always finite. Then, it is enough to choose the parameters in the bulk of order one, to obtain $\Omega_{bulk} \simeq \mathcal{O}(1)$. In order to obtain a hierarchy between bulk and cap volumes, it is necessary to demand that $\Omega_a \ll \mathcal{O}(1)$.

At this point we divide our discussion into two parts: first we consider the ‘special case’ cap solution discussed in section 4.1, whose simplicity allows simple explicit solutions. We then discuss the same question in the more general case.

The special case. Using the solutions found in the ‘special case’ section together with the continuity equation (3.4), we may evaluate the cap volume, eq. (4.52), in terms of bulk parameters:

$$\Omega_a \simeq 2\pi \left(\frac{4\lambda_2 \lambda_1^4 g_a}{g q^4} \right)^{1/2} \exp \left[\omega - 2\lambda_1 \xi_1 + (\lambda_3 + 2\lambda_1) \eta_a \right]. \tag{4.54}$$

Now, since the coefficient of η_a in these expressions is positive, it is clear that taking η_a large and negative corresponds here to making Ω_a small. Also, from eqs. (4.6) and (4.8) we see that the hierarchy assumptions, eq. (4.51), are easily satisfied in the limit we consider. Thus, we were indeed justified in using the asymptotic form for the hyperbolic functions.

But for the assumptions of the ‘special case’ model, matching also gives the value of η_a as

$$(\lambda_2 - \lambda_1) \eta_a = \lambda_1 \xi_1 - \lambda_2 \xi_2 + \ln \left(\frac{2\lambda_1 g_a}{q} \right), \tag{4.55}$$

which shows that η_a can be made large and negative if we take g_a to be small. Then equation (4.12) shows that this condition can be achieved by choosing the bulk parameters such that

$$\frac{2k_a}{N} \simeq 1 + \frac{\lambda_3}{2\lambda_1}. \tag{4.56}$$

If this condition is satisfied, then the volume of cap a is small. Analogous considerations for cap b give similar results.

The general case. We now evaluate the cap volume using the general solutions found earlier. If we also use the hierarchy assumptions, eq. (4.51), and the continuity equation (3.4), we calculate the cap volume to be

$$\begin{aligned} \Omega_a &\simeq 2\pi \left(\frac{g \lambda_1 \cosh \Delta_{2a}}{|q| \lambda_2 \cosh \Delta_{1a}} \right) (\mathcal{A}^2 \mathcal{W}^4)|_{\eta_a} \\ &= \pi \mathcal{A}^2|_{\eta_a}. \end{aligned} \tag{4.57}$$

For the generic situation of $\mathcal{O}(1)$ bulk parameters, we see from eq. (2.5) that $\mathcal{A}^2|_{\eta_a} \ll 1$ in the limit of large $|\eta_a|$. Thus, we obtain the desired result: $\Omega_a \ll \Omega_{\text{bulk}} \sim \mathcal{O}(1)$. Alternatively, if we instead wish to have cap volumes which are $\mathcal{O}(1)$ and bulk volumes which are much larger, we simply need to choose $\omega \gg 1$ while keeping all other bulk parameters fixed.

It remains now to show what conditions must be imposed on the bulk parameters and cap parameters in order to ensure that conditions (4.51) are satisfied. To simplify this discussion, we only consider the case $\lambda_3 = 0$. We accomplish this by adjusting the background gauge coupling, e , so that it is approximately equal to its value, $e_0 = qN/(2\lambda_1)$, in the absence of caps. More precisely, if we define

$$\epsilon = \frac{1}{2} e^{-\lambda_1(\eta_a - \xi_1)} \left[\frac{qN}{\lambda_1 e} - 2 \right], \tag{4.58}$$

then we should take

$$\epsilon \ll 1 \quad \text{and} \quad \Lambda_{1b} \gg 1 \tag{4.59}$$

and, for definiteness, take $\eta_a \approx -\eta_b$. In this case, the general cap solutions found earlier satisfy the desired hierarchy conditions (4.51). The analogous hierarchy conditions at brane b are much simpler to satisfy due to the fact that we get to choose freely Λ_{1b} . For example, choosing η_b large and $\Lambda_{1b} \sim \Delta_{1b} \gg 1$ guarantees that $\Lambda_{2b} \gg 1$ and so the two hierarchy constraints are satisfied.

To summarize, we see here how to obtain regularizing caps which are much smaller than the bulk volume, by appropriately tuning the gauge coupling e and by choosing large coordinate values for the brane positions. We have also shown that requiring such a hierarchy at only a single brane is not difficult to achieve in the sense that it involves no tuning of any bulk parameters.

4.4 Low-energy 4D effective potential

We next dimensionally reduce the capped bulk to 4 dimensions in order to identify more explicitly how 4-brane action influences the stabilization of the would-be flat direction parameterized by ω . In this section we restrict ourselves to evaluating the effective 4D potential for ω within the classical approximation.

To this end we identify the effective 4D action $S_{\text{eff}} = \int d^4x \mathcal{L}_{\text{eff}}$ by computing the 6D action at a one-parameter family of classical solutions labelled by the constant ω :

$$S_{\text{eff}} = \mathcal{S}_a + \mathcal{S}_b + S_{\text{cap } a} + S_{\text{bulk}} + S_{\text{cap } b}, \quad (4.60)$$

where $\mathcal{S}_a = \int d^4x \mathcal{L}_a$ and $\mathcal{S}_b = \int d^4x \mathcal{L}_b$ represent the 4-brane action for caps a and b , given by eq. (3.11), while $S_M = \int_M d^6x \mathcal{L} + S_{\text{GH}}(\partial M)$ represents the 6D bulk action, including the Gibbons-Hawking boundary contribution, defined by eqs. (2.1) and (2.3). The three last terms correspond to dividing the integration over the 2 extra dimensions into the three intervals defining the bulk, cap a or cap b .

Following [11], we see that using the 6D field equations, (2.2), to simplify the 6D bulk action in a region M with boundaries leads to the simple expression (with $\kappa^2 = 1$)

$$S_{\text{cl}} = \frac{1}{2} \int_M d^6x \sqrt{-g} \square \phi_{\text{cl}} - \int_{\partial M} d^5x \sqrt{-\gamma} K_{\text{cl}}, \quad (4.61)$$

which, together with Gauss' Law, allows the last three terms in eq. (4.60) to be written

$$S_{\text{cap } a} + S_{\text{bulk}} + S_{\text{cap } b} = -\frac{1}{2} \int d^5x \left([\sqrt{-g} \partial^\eta \phi + 2\sqrt{-\gamma} K]_{\eta_a} + [\sqrt{-g} \partial^\eta \phi + 2\sqrt{-\gamma} K]_{\eta_b} \right), \quad (4.62)$$

where as before $[f(\eta)]_{\eta_a} = f(\eta_a + \epsilon) - f(\eta_a - \epsilon)$ (and similarly for η_b).

Writing $S_{\text{eff}} = \int d^4x \mathcal{L}_{\text{eff}}$, and evaluating the right-hand-side of this last expression using the Israel and dilaton jump conditions, (3.21), (3.22) and (3.26) finally gives

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= 2\pi \sum_{i=a,b} \mathcal{A} \mathcal{W}^4 e^{2(\omega-p)} \left[\left(-V_i + \frac{5V_i}{4} - \frac{1}{2} \frac{dV_i}{d\phi} \right) + \frac{1}{2\mathcal{A}^2} \left(-U_i + \frac{3U_i}{4} - \frac{1}{2} \frac{dU_i}{d\phi} \right) (k_i - eA_\psi)^2 \right] \\ &= \pi \sum_{i=a,b} \mathcal{A} \mathcal{W}^4 e^{2(\omega-p)} \left[\left(\frac{V_i}{2} - \frac{dV_i}{d\phi} \right) - \frac{1}{2\mathcal{A}^2} \left(\frac{U_i}{2} + \frac{dU_i}{d\phi} \right) (k_i - eA_\psi)^2 \right]. \end{aligned} \quad (4.63)$$

Finally, to make the ω -dependence explicit we write $\mathcal{A} = \mathcal{A}_0 e^{\omega/2}$, $\phi = \phi_0 - \omega$, and choose for concreteness $V(\phi) = v e^{s\phi}$ and $U(\phi) = u e^{t\phi}$. Identifying $V_{\text{eff}} = -\mathcal{L}_{\text{eff}}$, we find

$$V_{\text{eff}}(\omega) = \sum_{i=a,b} \left[C_{V_i} e^{(5/2-s_i)\omega} + C_{U_i} e^{(3/2-t_i)\omega} \right], \quad (4.64)$$

where

$$\begin{aligned} C_{V_i} &= \pi \left[\mathcal{A}_0 \mathcal{W}^4 \left(\frac{1}{2} - s_i \right) v_i e^{s_i \phi_0 - 2p} \right]_{\eta=\eta_i} \\ C_{U_i} &= -\frac{\pi}{2} \left[\frac{\mathcal{W}^4}{\mathcal{A}_0} \left(\frac{1}{2} + t_i \right) u_i e^{t_i \phi_0 - 2p} (k_i - eA_\psi)^2 \right]_{\eta=\eta_i}. \end{aligned} \quad (4.65)$$

It is clear that this potential generically only has runaway solutions when both C_{U_i} and C_{V_i} and both of the coefficients of ω in the exponents have the same sign, but has nontrivial minima when some of these signs differ. Given the explicit relative sign appearing in eqs. (4.65), and positive u_i and v_i , we expect that stabilization of ω to be fairly generic.

The scale invariant case. Of particular interest is the case of scale-invariant branes, for which we have $s_i = 1/2$ and $t_i = -1/2$. In this case, not only do we recover the generic scale-invariant form for the potential

$$V_{\text{eff}}(\omega) = C e^{2\omega} \quad \text{with} \quad C = \sum_{i=a,b} (C_{U_i} + C_{V_i}), \quad (4.66)$$

but we also learn that $C = C_{U_i} = C_{V_i} = 0$. This agrees, and makes more precise, the arguments of ref. [11], wherein the same conclusion was drawn when scale-invariant branes were characterized as delta-function sources.

5. Conclusions

In this paper we present a regularization procedure for resolving the singularities in the most general axially symmetric, 4D-flat solutions to 6D gauged, chiral supergravity. This procedure resolves the singularities of these geometries using an explicit, but broad, class of cylindrical 4-branes that couple with the bulk Maxwell, dilaton and gravitational fields. The space interior to these 4-branes is capped off using the most general smooth, 4D-flat, and axially symmetric solutions to the same 6D supergravity equations that were used in the bulk between the two branes. Our analysis provides the necessary tools required to precisely explore the connections between properties of the bulk field configurations and the structure of the branes which source them.

We keep our analysis very general, with the goal of being able to map out these connections with as few restrictions as possible. We show, in particular, that the class of caps and 4-brane actions we consider contain sufficient numbers of parameters to cap an arbitrary axially-symmetric and 4D-flat bulk geometry. We also show that once the properties of one of the 4-brane caps is specified, there are sufficient parameters in the bulk geometry and in the other cap to complete the geometry. This both identifies the properties of the bulk sourced by a given brane, and precisely identifies how the properties of the brane at the other end of the bulk are dictated by those of the source brane with which one starts.

Knowing the properties of the caps shows that the presence of regularizing branes has important consequences on the properties of the bulk solutions. In particular, we show how the classical degeneracy amongst bulk geometries having different volumes can be lifted by the coupling of the 4-branes with the 6D dilaton. This provides a stabilization mechanism for the bulk, which relates the size of the extra dimensions with brane properties. By performing a dimensional reduction we also identify the effective 4D potential which captures this stabilization mechanism in the low-energy limit. We are able to do because our regulated 6D configurations are smooth everywhere, with the bulk fields not diverging at the brane positions (as they do for the effective co-dimension two 3-branes obtained in the thin-brane limit when the circumference of the 4-brane is taken to zero).

There are several directions in which our geometrical construction of the regularizing caps might be extended. First, the form of 4-brane action considered could be further generalized, such as by depending on additional brane-localized fields. The back-reaction

of such fields on the bulk configuration could then be consistently taken into account by studying their effects on the continuity and junction conditions. An important special case along these lines consists of studying the effects of integrating out massive brane fields, to see how this affects the condition of 4D flatness. Work along these lines is currently in progress [39].

Alternatively, our analysis could also be extended by generalizing the class of bulk configurations for which caps can be constructed. Of particular interest is such an extension to bulk geometries which are not 4D flat [30], for which one might imagine using regulating cap geometries which are less symmetric than the ones we consider here. Alternatively, extensions to configurations in more than six dimensions are also of interest, since bulk fields generically diverge at brane positions in this case as well.

Such constructions would be particularly useful for identifying more precisely how the cosmological constant problem gets rephrased in its extra-dimensional context. For these purposes it is important to be able to find regularizing caps that are general enough to characterize a large class of bulk geometries, in order to explore all of the naturalness issues which might be associated with a given regularization procedure.

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